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Remarks on Lewenstein-Sanpera Decomposition

Mingjun Shi*, Jiangfeng Du
Laboratory of Quantum Communication and Quantum Computation,
Department of Modern Physics,
University of Science and Technology of China, Hefei, 230026, P.R.China

Abstract

We discuss in this letter Lewenstein-Sanpera (L-S) decomposition for a specific Werner state. Compared with the optimal case, we propose a quasi-optimal one which in the view of concurrence leads to the same entanglement measure for the entangled mixed state discussed. We think that in order to obtain entanglement of given state the optimal L-S decomposition is not necessary.

*Email: shmj@ustc.edu.cn

Non-locality, entanglement or inseparability are some of the most genuine quantum concepts,[1,2,3] and play a very important role in quantum computation and quantum communication.[4,5,6] To study the entanglement phenomena embodied in mixed states is an intricate work. Associated with different definitions of the entanglement of mixed states, various quantitative measures have been proposed.[6,7,8,9,10] Among them is Lewenstein-Sanpera (L-S) decomposition.[9] It is said in Ref[9] that any density matrix ρ in $\mathbb{C}^2 \times \mathbb{C}^2$ can be decomposed as

$$\rho = \lambda \rho_s + (1 - \lambda) P_e, \quad \lambda \in [0, 1], \tag{1}$$

where ρ_s is separable density matrix, P_e denotes a single pure entangled projector ($P_e \equiv |\Psi_e\rangle \langle \Psi_e|$). Given ρ there are many different ρ_s's and P_e's satisfying Eqn.(1). The optimal case is unique in which λ is maximal, that is

$$\rho = \lambda^{(opt)} \rho_s^{(opt)} + (1 - \lambda^{(opt)}) P_e^{(opt)}.$$

Any other decomposition of the form $\rho = \widetilde{\lambda} \widetilde{\rho_s} + \left(1 - \widetilde{\lambda}\right) \widetilde{P_e}$, with $\widetilde{\lambda} \in [0, 1]$ such that $\widetilde{\rho_s} \neq \rho_s^{(opt)}$ necessarily implies that $\widetilde{\lambda} < \lambda^{(opt)}$. Then due to the uniqueness, the decomposition given by Eqn.(1) leads to an unambiguous measure of the entanglement for any mixed state ρ, that is

$$E(\rho) = (1 - \lambda^{(opt)}) \times E(|\Psi_e\rangle^{(opt)}), \tag{2}$$

where $E(|\Psi_e\rangle^{(opt)})$ is the entanglement of its pure state expressed in terms of the von Neumann entropy of reduced density matrix of either of its subsystems. And moreover, L-S point out that this measure of entanglement is independent of any purification or formation procedure.

In this letter, we discuss L-S decomposition for a specific entangled Werner's state. The optimal decomposition is discussed in Ref[9] and Ref,[11] where λ in Eqn.(1) reaches the maximal $\lambda^{(opt)}$ while $|\Psi_e\rangle^{(opt)}$ denotes the maximal entangled state, Bell state. But we will be interested in the quasi-optimal decomposition in which $|\Psi_e\rangle$ is not Bell state and λ is maximal relative to the $|\Psi_e\rangle$. We find that on the basis of entanglement concurrence[8] the optimal decomposition and the quasi-optimal one give the same result. The detail is given as follows.

We have known that a Werner's state can be expressed as

$$\rho_w = (1 - \epsilon) \rho_0 + \epsilon P_{Bell}, \quad \epsilon \in [0, 1], \tag{3}$$

where ρ_0 is the maximal separable state, i.e., $\rho_0 = \frac{1}{4} I_{4\times 4}$, $P_{Bell} = |\Psi_{Bell}\rangle \langle \Psi_{Bell}|$, and $|\Psi_{Bell}\rangle$ is one of four Bell states. If and only if $\frac{1}{3} < \epsilon \leq 1$, ρ_w is inseparable. In this letter, we focus on a specific case in which $\epsilon = \frac{1}{2}$ and $|\Psi_{Bell}\rangle = |\Psi^-\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle)$. That is,

$$\rho_{\frac{1}{2}} = \frac{1}{2} \rho_0 + \frac{1}{2} |\Psi^-\rangle \langle \Psi^-| = \frac{1}{4} \left(I \otimes I - \frac{1}{2} \sigma_1 \otimes \sigma_1 - \frac{1}{2} \sigma_2 \otimes \sigma_2 - \frac{1}{2} \sigma_3 \otimes \sigma_3 \right), \tag{4}$$

where I is 2×2 identity matrix, and σ_i is pauli matrix.

The optimal L-S decomposition has been obtained in Ref,[11] that is, for $\rho_{\frac{1}{2}}$ expressed by (4), $\lambda^{(opt)}=\frac{3}{4}$, $P_e^{(opt)}=\left|\Psi^-\right\rangle\left\langle\Psi^-\right|$, and $\rho_{\frac{1}{2}}$ can be written as

$$\rho_{\frac{1}{2}}=\frac{3}{4}\rho_s^{(opt)}+\frac{1}{4}\left|\Psi^-\right\rangle\left\langle\Psi^-\right|, \tag{5}$$

where $\rho_s^{(opt)}=\frac{2}{3}\rho_0+\frac{1}{3}\left|\Psi^-\right\rangle\left\langle\Psi^-\right|$, and evidently $\rho_s^{(opt)}$ is separable. According to Eqn.(2), the entanglement of $\rho_{\frac{1}{2}}$ is

$$E\left(\rho_{\frac{1}{2}}\right)=\frac{1}{4}E(\left|\Psi^-\right\rangle)=\frac{1}{4}. \tag{6}$$

Now we consider a decomposition of a more general form,

$$\rho_{\frac{1}{2}}=\delta\rho_s+(1-\delta)\left|\Psi\right\rangle\left\langle\Psi\right|,\quad \delta\in[0,1], \tag{7}$$

where we assume $\left|\Psi\right\rangle=\cos\theta\left|01\right\rangle-\sin\theta\left\langle 10\right|,\ \theta\in\left[0,\frac{\pi}{2}\right]$. We wish to find the maximal δ relative to $\left|\Psi\right\rangle$. This decomposition can be called quasi-optimal decomposition.

Let $x=\frac{1}{\delta}-1$. x is non-negative. We rewrite Eqn.(7) as

$$\rho_s=(1+x)\,\rho_{\frac{1}{2}}-x\left|\Psi\right\rangle\left\langle\Psi\right|,\quad x>0. \tag{8}$$

In other words, ρ_s is the pseudo-mixture of $\rho_{\frac{1}{2}}$ and $\left|\Psi\right\rangle\left\langle\Psi\right|$. Furthermore, we give the detailed form of Eqn.(8).

$$\begin{aligned}\rho_s \;=\;& \frac{1}{4}[I\otimes I-x\cos 2\theta\ \sigma_3\otimes I+x\cos 2\theta\ I\otimes\sigma_3\\ &+\left(x\sin 2\theta-\frac{1}{2}\left(1+x\right)\right)\sigma_1\otimes\sigma_1\\ &+\left(x\sin 2\theta-\frac{1}{2}\left(1+x\right)\right)\sigma_2\otimes\sigma_2\\ &-\frac{1}{2}\left(1-x\right)\sigma_3\otimes\sigma_3].\end{aligned} \tag{9}$$

Then the problem is to find the minimal x such that ρ_s is a separable state. To guarantee the positivity of ρ_s, we should know the eigenvalues of ρ_s. Or we can use the positivity criteria in Ref.[11] To determine the separability of ρ_s, we shall consider the positivity of the partial transposition of ρ_s ($\rho_s^{T_A}$ or $\rho_s^{Y_B}$) or the partial time-reversal of ρ_s ($\widetilde{\rho_s}$).[12,13,14,15] Again the criteria in Ref[11] can be used to verify the positivity of $\widetilde{\rho_s}$. Through laborious but not difficult mathematical computation we have the following results:

(i) Decomposition of the form (7) can be accomplished only if $\sin 2\theta\geq\frac{7}{12}$. When $\sin 2\theta<\frac{7}{12}$, ρ_s expressed by (8) is either non-positive or inseparable.

(ii) Under the condition $\sin 2\theta \geq \frac{7}{12}$ satisfied, the minimal x which ensures positivity and separability of ρ_s in Eqn.(9) is

$$x_{min} = \frac{1}{4\sin 2\theta - 1}. \tag{10}$$

Correspondingly, the maximal δ is

$$\delta_{max} = 1 - \frac{1}{4\sin 2\theta}. \tag{11}$$

The δ_{max} is maximal relative to a proper entangled state $|\Psi\rangle$ and is just what we want to know to realize the quasi-optimal decomposition of $\rho_{\frac{1}{2}}$ in the form Eqn.(7).

Now let's discuss our results. First, in general sense, an arbitrary entangled state can not necessarily be used as the component of L-S decomposition of Eqn.(1). From the specific example discussed in this letter, we see that the entanglement of the pure state appearing in the decomposition can not be too small. Then, recall that the concept of concurrence for pure state.[8] For any pure state in $\mathbb{C}^2 \times \mathbb{C}^2$ described as

$$|\psi\rangle = c_0\,|00\rangle + c_1\,|01\rangle + c_2\,|10\rangle + c_3\,|11\rangle\,, \tag{12}$$

where c_i's are complex numbers and satisfies $\sum_{i=0}^{3} |c_i|^2 = 1$, the concurrence is defined as

$$C\left(|\psi\rangle\right) = 2\,|c_0c_3 - c_1c_2|\,. \tag{13}$$

And the entanglement can be expressed in terms of $C\left(|\psi\rangle\right)$, i.e.,

$$E\left(|\psi\rangle\right) = -\frac{1+\sqrt{1-C^2}}{2}\log_2\frac{1+\sqrt{1-C^2}}{2} - \frac{1-\sqrt{1-C^2}}{2}\log_2\frac{1-\sqrt{1-C^2}}{2} \tag{14}$$

The concurrence of Bell state $|\Psi^-\rangle$ is $C\left(|\Psi^-\rangle\right) = 1$, and that of $|\Psi\rangle$ in Eqn.(7) is $C\left(|\Psi\rangle\right) = 2\sin\theta\cos\theta = \sin 2\theta$. So for the optimal L-S decomposition of $\rho_{\frac{1}{2}}$, we have

$$\left(1-\lambda^{(opt)}\right) C\left(\left|\Psi^-\right\rangle\right) = \frac{1}{4}. \tag{15}$$

For quasi-optimal decomposition of $\rho_{\frac{1}{2}}$ with the form (7), we have

$$\left(1-\delta_{max}\right) C\left(|\Psi\rangle\right) = \left[1 - (1 - \frac{1}{4\sin 2\theta})\right]\sin 2\theta = \frac{1}{4}. \tag{16}$$

The same result of (15) and (16) means that at least for this specific $\rho_{\frac{1}{2}}$, the optimal and quasi-optimal decomposition can be used to demonstrate the entanglement proportion embodied in $\rho_{\frac{1}{2}}$ in terms of concurrence. Because of the

uniqueness, the optimal L-S decomposition indicates more strict constraints, and it is a hard work to find it. Comparatively speaking, the quasi-optimal decomposition is easy to realize and may be a convenient method to study entangled mixed states. On the other hand, we easily see that for $\rho_{\frac{1}{2}}$

$$\left(1-\lambda^{(opt)}\right) E\left(\left|\Psi^{-}\right\rangle\right)=\frac{1}{4} \neq\left(1-\delta_{max}\right) E\left(\left|\Psi\right\rangle\right). \tag{17}$$

That is, in term of von Neumann entropy, there is inconsistence between the optimal and the quasi-optimal. Note Eqn.(14). $E\left(\left|\psi\right\rangle\right)$ is a logarithmic function of C. We consider that logarithm conceals the agreement on the level of concurrence. So we think that concurrence is the proper quantity to measure the entanglement of mixed states in the frame of L-S decomposition. In our view, obtaining the quasi-optimal decomposition is sufficient to measure the entanglement proportion in a mixed state.

Of course, in this letter we have only studied a specific example. We wish to extend our discussion to more general cases. Further results will be submitted later.

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